Last Lecture:

each broken symmetry $\longrightarrow$ massless, spinless Goldstone boson

For a spin zero boson B of four-momentum p_B^μ, we have from Lorentz invariance:

$$\langle VAC | J^\lambda(x) | B \rangle = i \frac{F p_B^\lambda e^{ip_B \cdot x}}{(2\pi)^{3/2}\sqrt{2p_B^0}} \qquad (1)$$

where F is a constant

$\longrightarrow$ consistent with current conservation: $p_{B\lambda} p_B^\lambda = 0$

Further,

$$\langle B | \Phi_n(y) | VAC \rangle = \frac{Z_n e^{-ip_B \cdot y}}{(2\pi)^{3/2}\sqrt{2p_B^0}} \qquad (2)$$

with Z_n dimensionless constant.

(1) + (2) then gives:

$$(2\pi)^{-3} i \rho_n(-p^2) p^\lambda \theta(p^0) = \int d^3p_B \langle VAC | J^\lambda(0) | B \rangle \langle B | \Phi_n(0) | VAC \rangle \times \delta^4(p - p_B)$$

$$= \delta(p^0 - |\vec{p}|)(2\pi)^{-3}(2p^0)^{-1} p^\lambda i F Z_n$$

$$= \theta(p^0)\,\delta(-p^2)(2\pi)^{-3} p^\lambda i F Z_n \,,$$

so $\rho_n(\mu^2) = F Z_n \,\delta(\mu^2)$

Comparing to the result from last lecture:

$$iFZ_n = -\sum_m t_{nm} \langle \phi_m(0) \rangle_{VAC} \qquad (3)$$

For several broken symmetries with generators t_a and currents J^μ_a, we have

$$\langle VAC | J^\lambda_a(x) | B_b \rangle = \frac{i F_{ab} p_B^\lambda e^{ip_B \cdot x}}{(2\pi)^{3/2}\sqrt{2p_B^0}} ,$$

$$\langle B_a | \phi_n(y) | VAC \rangle = \frac{Z_{an} e^{-ip_B \cdot y}}{(2\pi)^{3/2}\sqrt{2p_B^0}}$$

$$\longrightarrow \; i \sum_b F_{ab} Z_{bn} = -\sum_m [t_a]_{nm} \langle \phi_m(0) \rangle_{VAC} \qquad (4)$$

For instance, in the $O(N)$ example, setting

$$\bar{\Phi}_m \equiv \langle \phi_m(0) \rangle_{VAC} = v \delta_{m1}$$

$\rightarrow$ the $N-1$ broken symmetry generators t_a (with $a = 2 \cdots N$) can be defined as rotations in the 1-a plane.

$$\longrightarrow \quad [t_a]_{1a} = -[t_a]_{a1} = i$$

The $N-1$ Goldstone bosons transform under unbroken $O(N-1)$ symmetry!

$$\longrightarrow \; F_{ab} = \delta_{ab} F , \quad Z_{a1} = 0 , \quad Z_{ab} = Z \delta_{ab}$$

Then eq. (4) gives

$$F Z = v$$

Convention: $Z = 1$, $F = v$

$\rightarrow$ F is a measure of the strength of the symmetry breaking

We can expand fields ϕ_n as follows

$$\phi_n(x) = \sum_a Z_{ab}\, \pi_a(x) + \cdots ,$$

Goldstone boson

fields <u>not</u> creating Goldstone bosons

From (4) we then have

$$Z_{ab} = \sum_b F^{-1}_{ab} \left(i t_b \bar{\Phi}\right)_n$$

$\rightarrow$ effective interaction for Goldstone bosons

$$\mathcal{H}_{eff} = \frac{1}{N!}\, g_{a_1 \cdots a_N}\, \pi_{a_1} \cdots \pi_{a_N} ,$$

with

$$g_{a_1 \cdots a_N} = \sum_{b_1 \cdots b_N} F^{-1}_{a_1 b_1} \cdots F^{-1}_{a_N b_N} \left(i t_{b_1} \bar{\Phi}\right)_{n_1} \cdots \left(i t_{b} \bar{\Phi}\right)_{n_N} \times \left. \frac{\partial^N V(\phi)}{\partial \phi_{n_1} \cdots \partial \phi_{n_N}} \right|_{\phi = \bar{\Phi}}$$

§5.2 Spontaneously broken approximate Symmetries

Add symmetry breaking perturbation to action:

$$V(\phi) = V_0(\phi) + \underset{\substack{\uparrow \\ \text{pert.}}}{V_1(\phi)}$$

where V_0 satisfies

$$\sum_{nm} \frac{\partial V_0(\phi)}{\partial \phi_n} (t_\alpha)_{nm} \phi_m = 0$$

Suppose that $V_1(\phi)$ shifts the minimum of of $V_0(\phi)$ from ϕ_0 to $\bar{\phi} = \phi_0 + \underset{\substack{\uparrow \\ \text{small}}}{\phi_1}$

$$\longrightarrow \quad \left.\frac{\partial V(\phi)}{\partial \phi_n}\right|_{\phi = \phi_0 + \phi_1} = 0 \qquad (1)$$

Suppose both $\phi_1 \sim \mathcal{O}(\varepsilon)$, $V_1 \sim \mathcal{O}(\varepsilon)$ with $\varepsilon \ll 1$

Then we can expand (1) up to 1st order in ε:

$$\underbrace{\left.\frac{\partial V_0(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0}}_{=0} + \sum_m \left.\frac{\partial^2 V_0(\phi)}{\partial \phi_n \partial \phi_m}\right|_{\phi=\phi_0} \phi_{1\,m}$$

$$+ \left.\frac{\partial V_1(\phi)}{\partial \phi_n}\right|_{\phi=\phi_0} = 0 \qquad (2)$$

V_0 and ϕ_0 satisfy the condition

$$\sum_{nl} \frac{\partial^2 V_0(\phi)}{\partial \phi_n \, \partial \phi_m}\bigg|_{\phi=\phi_0} (t_\alpha)_{nl} \, \phi_{0l} = 0$$

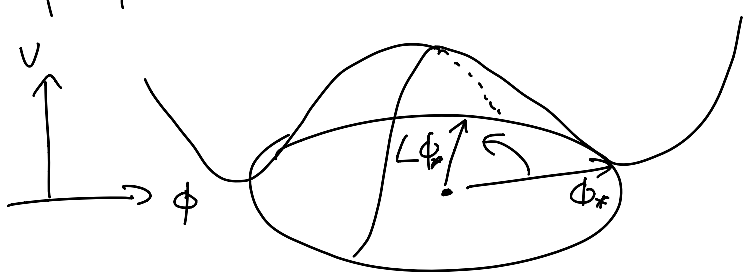
→ multiplying (2) with $(t_\alpha \phi_0)_n$ and summing over n gives

$$\sum_n (t_\alpha \phi_0)_n \frac{\partial V_1(\phi)}{\partial \phi_n}\bigg|_{\phi=\phi_0} = 0 \qquad (3)$$

Have to make sure eq. (3) is satisfied!

$V_0(\phi)$ is invariant under $\phi \mapsto L\phi$

→ if ϕ_* is minimum, then so is $L\phi_*$



Using
$$\frac{\partial L(\theta)}{\partial \theta^\alpha} L^{-1}(\theta) = i N_{\alpha\beta}(\theta) t_\beta ,$$
where $N_{\alpha\beta}$ is non-singular matrix depending on group parameters θ_α.

→ $V_1(L(\theta)\phi_*)$ must have a minimum at some θ_* with

$$0 = \frac{\partial V_1(L(\theta)\phi_*)}{\partial \theta_\alpha}\bigg|_{\theta=\theta_*} = \sum_n \frac{\partial V_1(\phi)}{\partial \phi_n}\bigg|_{\phi=L(\theta)\phi_*} N_{\alpha\beta}(\theta) (i t_\beta L(\theta)\phi_*)_n$$

Since $N_{\alpha\beta}$ is non-singular:

$$(4) \qquad 0 = \sum_n \frac{\partial V_1(\phi)}{\partial \phi_n}\bigg|_{\phi = L(\theta_*)\phi_*} (t_\beta L(\theta_*)\phi_*)_n$$

$\longrightarrow$ to satisfy equation (3) make the choice

$$\phi_0 = L(\theta_*)\,\phi_* \, .$$

"vacuum alignment" condition
ensures that the unbroken symmetry
remains the same
(for example $SO(N-1)$ in $O(N)$-models)

Now consider the mass-matrix

$$M^2_{ab} = \sum_{mn} Z_{an} Z_{bm} \frac{\partial^2 V(\phi)}{\partial \phi_m \partial \phi_n}\bigg|_{\phi = \phi_0 + \phi_1} ,$$

where Z_{an} is field renormalization constant.
expanding to 1st order gives:

$$M^2_{ab} = \sum_{mn} Z_{an} Z_{bm} \left[\sum_\ell \frac{\partial^3 V_0(\phi)}{\partial \phi_\ell \partial \phi_m \partial \phi_n}\bigg|_{\phi=\phi_0} \phi_{1\ell} + \frac{\partial^2 V_1(\phi)}{\partial \phi_m \partial \phi_n}\bigg|_{\phi=\phi_0} \right] \qquad (5)$$

(zeroth order term vanishes)

where

$$Z_{an} = \sum_b F^{-1}_{ab} \, (i t_b \phi_0)_n$$

Contracting with $(t_b \phi_0)_m \phi_{1l}$ gives

$$0 = \sum_{nml} \frac{\partial^3 V_0(\phi)}{\partial\phi_l \partial\phi_m \partial\phi_n}\bigg|_{\phi=\phi_0} \phi_{1l} (t_a \phi_0)_n (t_b \phi_0)_m$$

$$+ \sum_{nm} \frac{\partial^2 V_0(\phi)}{\partial\phi_n \partial\phi_m}\bigg|_{\phi=\phi_0} (t_a \phi_1)_n (t_b \phi_0)_m$$

$$+ \sum_{nl} \frac{\partial^2 V_0(\phi)}{\partial\phi_n \partial\phi_l}\bigg|_{\phi=\phi_0} (t_a t_b \phi_0)_n \phi_{1l} \qquad (6)$$

(exercise)

second term vanishes due to $\sum_{nm} \frac{\partial^2 V}{\partial\phi_n \partial\phi_l}\Big|_{\phi=\bar\phi} t_{nm}\bar\phi = 0$ and $\bar\phi = \phi_0$ here, the third can be rewritten using eq. (2)

$\rightarrow$ (6) becomes

$$\sum_{nml} \frac{\partial^3 V_0(\phi)}{\partial\phi_l \partial\phi_m \partial\phi_n}\bigg|_{\phi=\phi_0} \phi_{1l} (t_a \phi_0)_n (t_b \phi_0)_m = \frac{\partial V_1(\phi)}{\partial\phi_n}\bigg|_{\phi=\phi_0} (t_a t_b \phi_0)_n$$

$\rightarrow$ inserting back into (5) gives:

$$M_{cd}^2 = -\sum_{ab} F_{ca}^{-1} F_{db}^{-1} \left[(t_a \phi_0)_n (t_b \phi_0)_m \frac{\partial^2 V_1(\phi)}{\partial\phi_m \partial\phi_n}\bigg|_{\phi=\phi_0} + (t_a t_b \phi_0)_n \frac{\partial V_1(\phi)}{\partial\phi_n}\bigg|_{\phi=\phi_0} \right] \qquad (7)$$

$\rightarrow$ small but non-zero mass "pseudo Goldstone bosons"

Let's check that (7) gives a positive mass:

One can derive

$$M^2_{ab} = \sum_{cd\,\alpha\beta} N^{-1}_{a\alpha}(\theta_*)\, N^{-1}_{b\beta}(\theta_*)\, F^{-1}_{ac} F^{-1}_{bd} \underbrace{\left.\frac{\partial^2 V_1(L(\theta)\phi_*)}{\partial\theta_\alpha \partial\theta_\beta}\right|_{\theta=\theta_*}}_{\geq 0}$$

as θ_* is at minimum of $V_1(L(\theta)\phi_*)$

To see this, compute

$$\left.\frac{\partial^2 V_1(L(\theta)\phi_*)}{\partial\theta_\alpha\,\partial\theta_\beta}\right|_{\theta=\theta_*} = \frac{\partial}{\partial\theta_\beta}\left[\sum_n \left.\frac{\partial V_1(\phi)}{\partial\phi_n}\right|_{\phi=L(\theta)\phi_*} N_{\alpha\gamma}(\theta)(it_\gamma L(\theta)\phi_*)_n\right]_{\theta=\theta_*}$$

and set $\phi_0 = L(\theta_*)\phi_*$